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THE BANACH–MAZUR DISTANCE TO THE CUBE AND THE DVORETZKY–ROGERS FACTORIZATION

BY

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ABSTRACT

We show the existence of a sequence (λ_n) of scalars with $\lambda_n = o(n)$ such that, for any symmetric compact convex body $B \subset \mathbf{R}^n$, there is an affine transformation T satisfying $\mathbf{Q} \subset T(B) \subset \lambda_n \mathbf{Q}$, where $\mathbf{Q}$ is the n-dimensional cube. This complements results of the second-named author regarding the lower bound on such λ_n. We also show that if X is an n-dimensional Banach space and $m = [n/2]$, then there are operators $\alpha : l_2^m \to X$ and $\beta : X \to l_\infty^m$ with $\| \alpha \| \cdot \| \beta \| \leqq C$, where C is a universal constant; this may be called "the proportional Dvoretzky–Rogers factorization". These facts and their corollaries reveal new features of the structure of the Banach–Mazur compactum.

A celebrated result of F. John ([Joh]) says that, for every compact symmetric convex body $B \subset \mathbf{R}^n$, there is an affine transformation T such that $D \subset T(B) \subset n^{1/2}D$, where D is the Euclidean unit ball. In the language of the local theory of Banach spaces this means that, for *any* n-dimensional normed space X, *the Banach–Mazur distance* $d(X, l_2^n) \leqq n^{1/2}$, where

$$d(X, Y) \equiv \inf\{ \| T \| \cdot \| T^{-1} \| : T : X \to Y \text{ an isomorphism}\}.$$

The constant $n^{1/2}$ is clearly the best possible, as is seen by considering $X = l_\infty^n$ or l_1^n.

By contrast, no nontrivial information was available until very recently about similar problems with D replaced by other convex symmetric bodies,

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e.g. the unit balls of l_∞^n or l_1^n (for $n > 2$). The main purpose of this paper is to partially fill this gap by proving the following

THEOREM 1. *We have*

$$\max\{d(X, l_\infty^n) : \dim X = n\} = \max\{d(X, l_1^n) : \dim X = n\} = o(n).$$

Moreover, there is an explicit estimate from above of the form $Cn\exp(-c(\log n)^{1/2})$ *for the quantity in question, where* $C, c > 0$ *are universal constants.*

REMARKS. (1) The set $\mathcal{B} = \mathcal{B}_n \equiv \{X : X$ — normed space, $\dim X = n\}$, endowed with the Banach–Mazur "distance", is usually called *the Banach–Mazur compactum* (or *the Minkowski compactum*). The result of F. John mentioned above says that $\mathcal{B}_n$ is contained in the "circle" of radius $n^{1/2}$, centered at l_2^n. Theorem 1 states that for the center l_∞^n the corresponding radius is $o(n)$. We should recall here that, by the remarkable result of Gluskin [Glu], the diameter of $\mathcal{B}_n$ is asymptotically of order n and so, for *some* convex bodies, the trivial upper bound on the corresponding radius cannot be essentially improved.

(2) Our proof of Theorem 1 actually gives a somewhat more precise estimate: if $d_x \equiv d(X, l_2^n)$ ($\leqq n^{1/2}$), then

$$\max\{d(X, l_\infty^n), d(X, l_1^n)\} \leqq Cn^{1/2} d_x \exp(-c(\log d_x)^{1/2})$$

with $C, c > 0$ — universal constants. Moreover, for any $p \in [1, \infty]$, denoting $\alpha = |1/p - \frac{1}{2}|$ and $q = p/(p-1)$, we get, after routine changes in the argument,

$$d(X, l_p^n) \leqq Cn^\alpha d_x \exp(-c(\log(d_x/n^{1/q}))^{1/2}).$$

(3) The hypothesis of the symmetry of the convex body in question cannot be omitted. Indeed, a simple geometric argument shows that if Σ is an n-dimensional simplex, $K \subset \mathbf{R}^n$ — *any* symmetric convex body (in particular the Euclidean ball, cube or a ball in any l_p^n) and $K \subset \Sigma \subset \lambda K$, then $\lambda \geqq n$. We do not pursue the problem of asymptotics of the best λ in the nonsymmetric case as it lacks immediate motivation in the local theory of Banach spaces.

(4) The estimate $Cn\exp(-c(\log n)^{1/2})$ in Theorem 1 is apparently not sharp. To obtain a better one we would need sharper estimates on $b(\delta)$ in Proposition 3. However, one cannot hope for the same asymptotics $n^{1/2}$ as in the John's theorem: it was shown recently by the second-named author that $\max\{d(X, l_\infty^n) : X \in \mathcal{B}_n\} \geqq cn^{1/2} \log n$ ([Sza]).

Our next theorem improves the result of Dvoretzky–Rogers [D–R]. One may call it *"the proportional Dvoretzky–Rogers factorization"*.

THEOREM 2. *Given $\delta \in (0,1)$ there exists $A = A(\delta)$ such that, for any finite-dimensional Banach space X, there is an integer $m > (1-\delta)\dim X$ and vectors $x_1, \dots, x_m \in X$ such that, for any scalars $t_1, \dots, t_n$,*

$$\max_{j \leqq m} |t_j| \leqq \left\| \sum_{j \leqq m} t_j x_j \right\| \leqq A \left(\sum_{j \leqq m} |t_j|^2 \right)^{1/2}.$$

In other words, the formal identity $i_{2,\infty} : l_2^m \to l_\infty^m$ can be written as $i_{2,\infty} = \alpha \circ \beta$, $\beta : l_2^m \to X$, $\alpha : X \to l_\infty^m$, with $\| \alpha \| \cdot \| \beta \| \leqq A$. The same holds for the formal identity $i_{1,2} : l_1^m \to l_2^m$.

REMARK. (5) [D–R] contained a version of Theorem 2 with $m = [n^{1/2}]$. On the other hand, one cannot expect the statement to hold with $m = n$; it was shown in [Sza] that, in general, one may need then $A \geqq c(n/\log n)^{1/10}$. However, our methods do not seem to imply that, in the case $m = n$, $A = o(n^{1/2})$.

(6) A version of Theorem 2 with "given $\delta \in (0,1)$" replaced by "there exists $\delta \in (0,1)$" is much easier to prove. Indeed, one just needs to combine Lemmas B and E (both known) with the same argument as that used to derive Theorem 2 from Proposition 3 below.

(7) It follows immediately from Theorem 2 that, given $\delta \in (0,1)$, there is a subspace E of X with $\dim E = m \geqq (1-\delta)n$ and $d(E, l_\infty^m) \leqq A(\delta) n^{1/2}$ (resp. $d(E, l_1^m)$). In the case of l_1^m, this was proved, up to a logarithmic factor, in [B–M]; a version with "there exists δ" was shown, independently of this work and approximately at the same time, by K. Ball.

(8) The same argument as used below to prove Theorem 2 shows that if $\dim X = n$ and $m = [n/2]$, then the Banach–Mazur distance of $Y = l_2^m \oplus l_1^{n-m}$ (or $l_2^m \oplus l_\infty^{n-m}$) to X does not exceed $Cn^{1/2}$, C — a universal constant. This means that, in the asymptotic sense, Y is a "center" of the Banach–Mazur compactum, which answers a question of A. Pelczynski (stated e.g. in his 1983 Congress talk [Pel]) about uniqueness of such a center; note also that $d(Y, l_2^n) \geqq (\frac{1}{2}n)^{1/2}$. Actually, for this purpose the original Dvoretzky–Rogers lemma would suffice, but without yielding the extremal distance of a center to l_2^n.

We use the standard notation from the local theory of Banach spaces as can be found e.g. in [T–J] or [M–S]. Let us only mention the following: by $|\cdot|_p$ we denote the norm on l_p^n, by $|\cdot|$ — the norm on a generic Hilbert space (and also the cardinality of a set), by $\| T \|_{p \to q}$ — the norm of an operator T considered as acting from l_p^n to l_q^m; for a subspace E of a Hilbert space P_E is the orthogonal projection onto E. We consider only the real case; however, all the statements hold also in, and all the arguments carry over to, the complex case (the crucial Lemma B was stated in [B–T] in the complex version). The letters C, c, c_0, c', etc. are reserved for universal numerical constants, whose value may change from line to line.

The theorems will follow from the following technical statement which is a more precise version of Theorem 2 (but, in fact, is formally equivalent to it).

Proposition 3. *Given $\delta \in (0, 1)$ there exists $b = b(\delta) > 0$ such that, for any finite-dimensional normed space X, there is a Euclidean norm $|\cdot|$ on X, an integer $m \geqq (1-\delta)\dim X$ and $x_1, \ldots, x_m \in X$, $\| x_j \| \leqq 1$ for all j, such that*

(1) $|\cdot| \leqq \|\cdot\|_X \leqq 2^{1/2} d_x |\cdot|$, *where* $d_x = d(X, l_2^{\dim X})$,

(2) *for any scalars* $t_1, \ldots, t_m$,

$$b\left(\sum |t_j|^2\right)^{1/2} \leqq \left|\sum t_j x_j\right|.$$

Moreover, $b(\delta) \geqq c \exp(-C(\log 1/\delta)^2)$, where c, $C > 0$ are universal constants.

We first show how to derive the Theorems from Proposition 3.

Proof of Theorem 1. We prove the stronger version indicated in the remark (2) following the Theorem. By duality, it is enough to estimate $d(X, l_1^n)$. Apply Proposition 3 to X (with δ to be specified later) to obtain Euclidean structure on X and $x_1, \ldots, x_m$. Let $y_1, \ldots, y_{n-m}$ be an orthogonal basis of the orthogonal complement of $[x_1, \ldots, x_m]$ (the linear span of x_j's), normalized so that $|y_j| = (2^{1/2} d_x)^{-1}$ and so $\| y_j \| \leqq 1$ for all j. Consequently, for any sequences of scalars (t_j) and (s_i),

$$\left\| \sum t_j x_j + \sum s_i y_i \right\| \leqq \sum |t_j| + \sum |s_i|.$$

On the other hand

$$\begin{aligned}\sum |t_j| + \sum |s_i| &\leqq m^{1/2}\left(\sum |t_j|^2\right)^{1/2} + (n-m)^{1/2}\left(\sum |s_i|^2\right)^{1/2} \\ &\leqq m^{1/2}b(\delta)^{-1}\left|\sum t_jx_j\right| + (n-m)^{1/2}\cdot 2^{1/2}d_x\left|\sum s_iy_i\right| \\ &\leqq (mb(\delta)^{-2} + 2(n-m)d_x^2)^{1/2}\left|\sum t_jx_j + \sum s_iy_i\right| \\ &\leqq (nb(\delta)^{-2} + 2\delta nd_x^2)^{1/2}\left\|\sum t_jx_j + \sum s_iy_i\right\|.\end{aligned}$$

To conclude the argument use the estimate on $b(\delta)$ given in Proposition 3 and optimize over $\delta\in(0, 1)$. Let us also note that the case of the general p (from the remark (2)) requires only standard modifications and so we omit the proof.

Proof of Theorem 2. First observe that of the three versions given in the statement of the Theorem, the first two are equivalent by the extension property of the space l_∞ and the last two by duality. We will show the last statement (about the factorization of $i_{1,2}$ through X). Applying Proposition 3 (with the same δ), defining $\beta: l_1^m\to X$ by $\beta(t_j)=\sum t_jx_j$, $\alpha_0:[x_j]\to l_2^m$ by $\alpha_0(\sum t_jx_j)=(t_j)$ and finally $\alpha: X\to l_2^m$ by $\alpha=\alpha_0P$, where P is the orthogonal projection of l_2^n onto $[x_j]$, we get the required factorization $i_{1,2}=\alpha\circ\beta$.

For the proof of Proposition 3 we need several lemmas.

Lemma A. *Let $A=(a_{ij})$ be a triangular $k\times k$ matrix such that, for some $M>c>0$,*
(i) $\|A\|_{2\to 2}\leqq M$,
(ii) $|a_{ii}|\geqq c$ *for* $i=1,\dots,k$.
Then $\|A^{-1}\|_{2\to 2}\leqq 2c^{-1}(M/c+1)^{k-1}$.

Proof. Denote the diagonal and the off-diagonal parts of A by Δ and T respectively. Then $A=\Delta(I+\Delta^{-1}T)$, $A^{-1}=(I+\Delta^{-1}T)^{-1}\Delta^{-1}$; the argument is concluded by expanding $(I+\Delta^{-1}T)^{-1}$ into a (terminating after k terms) power series and applying all the available estimates.

The next lemma, which is the cornerstone of this paper, is a special case of Theorem 7.1 from [B–T]. We reproduce the proof here for completeness as it becomes significantly simpler now than in the more general setting of [B–T].

Lemma B. *Let $x_1,\dots,x_n\in l_2$ and $\alpha>0$ be such that*
(i) $|x_j|_2\leqq 1$ *for all* j.

(ii) $|\langle x_j, e_j\rangle| \geqq \alpha$ *for all* j.

Then there exists $\sigma \subset \{1, 2, \dots, n\}$, $|\sigma| > cn$, *such that, for all scalars* (t_j),

$$\left| \sum t_j x_j \right|_2 \geqq \tfrac{1}{4}\alpha \left(\sum |t_j|^2 \right)^{1/2},$$

where $c = c(\alpha)$ *depends only on* α.

PROOF. *Step I.* We show now that given $\delta > 0$ (to be specified later), there exists $\sigma_1 \subset \{1, \dots, n\}$, $|\sigma_1| = n_1 > \frac{1}{4}\delta n$, such that, denoting $a_{ij} = \langle x_i, e_j\rangle$, we have, for $i \in \sigma_1$,

$$(*) \qquad \left(\sum_{j \in \sigma_1, j \neq i} |a_{ij}|^2 \right)^{1/2} \leqq \delta.$$

We present an argument due to J. Elton (see e.g. [J–S]). Let $k = [\frac{1}{2}\delta^2 n]$, we then have

$$\operatorname*{Ave}_{\substack{I \subset \{1, \dots, n\}, \\ |I| = k}} \sum_{i, j \in I, j \neq i} |a_{ij}|^2 = \binom{n-2}{k-2} \Big/ \binom{n}{k} \sum_{1 \leqq i \neq j \leqq n} |a_{ij}|^2 \leqq \binom{n-2}{k-2} \Big/ \binom{n}{k} \cdot n$$

$$= k(k-1)/(n-1) \leqq k^2/n.$$

Choose I_0, for which $\Sigma_{i,j \in I_0, i \neq j} |a_{ij}|^2 \leqq k^2/n$, and define $\sigma_1 = \{i \in I_0 : \Sigma_{j \in I_0, j \neq i} |a_{ij}|^2 \leqq 2k/n\}$; then $|\sigma_1| > \frac{1}{2}k$. One easily checks that this choice of σ_1 works.

Step II. Assume, for simplicity, that $\sigma_1 = \{1, 2, \dots, n_1\}$. Denote, for $i < n_1$, $y_i = \Sigma_{j \leqq n_1} a_{ij} e_j$ and $y'_i = \Sigma_{j \leqq n_1, j \neq i} a_{ij} e_j$. By Step I, $|y'_i|_2 \leqq \delta$ for $i \leqq n_1$ and consequently, denoting

$$\mathscr{D} = \left\{ \varepsilon = \sum_{j \leqq n_1} \varepsilon_j e_j : \varepsilon_j = +1 \text{ or } -1 \right\},$$

we have

$$\operatorname*{Ave}_{\varepsilon \in \mathscr{D}} \left(\sum_i |\langle y'_i, \varepsilon\rangle|^2 \right)^{1/2} \leqq \left(\operatorname*{Ave}_{\varepsilon \in \mathscr{D}} \sum_i |\langle y'_i, \varepsilon\rangle|^2 \right)^{1/2} = \left(\sum_i |y'_i|_2^2 \right)^{1/2} \leqq \delta n_1^{1/2}.$$

Now let $\mathscr{S} = \{\varepsilon \in \mathscr{D} : \Sigma_i |\langle y'_i, \varepsilon\rangle|^2)^{1/2} \leqq 2\delta n_1^{1/2}\}$, then $|\mathscr{S}| > 2^{n_1 - 1} = \frac{1}{2}|\mathscr{D}|$ and so, by the Sauer–Shelah lemma ([Sau], [She]), there is $\sigma_2 \subset \{1, 2, \dots, n_1\} = \sigma_1$, $|\sigma_2| = n_2 > \frac{1}{2}n_1 \geqq \frac{1}{8}\delta^2 n$ such that, for each $\varepsilon' = (\varepsilon'_j)_{j \in \sigma_2} \in \{-1, 1\}^{\sigma_2}$, there exists an extension $\varepsilon = (\varepsilon_j)_{j \in \sigma_1} \in \mathscr{S}$ (i.e. $\varepsilon_j = \varepsilon'_j$ if $j \in \sigma_2$).

Step III. We claim that, for any scalars $(a_i)_{i\in\sigma_2}$,

$$\left|\sum_i a_i y_i\right|_1 \geqq \alpha \sum_i |a_i| - 2^{3/2}\delta n_2^{1/2}\left(\sum_i |a_i|^2\right)^{1/2}.$$

Indeed, set, for $i\in\sigma_2$, $\varepsilon'_i = \operatorname{sgn}(a_{ii}\cdot a_i)$ (recall that $a_{ii} = \langle x_i, e_i\rangle = \langle y_i, e_i\rangle$, $|a_{ii}| \geqq \alpha$) and let $\varepsilon = (\varepsilon_j)_{j\in\sigma_1}\in\mathcal{S}$ be an extension of $\varepsilon' = (\varepsilon'_i)$, the existence of which is guaranteed by Step II. Then

$$\left|\sum_{i\in\sigma_2} a_i y_i\right|_1 \geqq \left\langle\sum_i a_i y_i, \varepsilon\right\rangle = \sum_i |a_i|\cdot|a_{ii}| + \left\langle\sum_i a_i y'_i, \varepsilon\right\rangle$$

$$\geqq \alpha\sum_i |a_i| - \left(\sum_i |\langle y_i, \varepsilon\rangle|^2\right)^{1/2}\left(\sum_i |a_i|^2\right)^{1/2} \geqq \alpha\sum_i |a_i| - 2\delta n_1^{1/2}\left(\sum_i |a_i|^2\right)^{1/2},$$

whence our claim promptly follows.

Step IV. We show that, with proper choice of δ, there exists $\sigma\subset\sigma_2$, $|\sigma| \geqq \frac{1}{2}|\sigma_2| \geqq 2^{-4}\delta^2 n$, so that, for all scalars $(t_i)_{i\in\sigma}$,

$$\left|\sum_{i\in\sigma} t_i y_i\right|_2 \geqq \tfrac{1}{4}\alpha\left(\sum_{i\in\sigma} |t_i|^2\right)^{1/2}.$$

From this the lemma easily follows as $|\Sigma\, t_i x_i|_2 \geqq |\Sigma\, t_i y_i|_2$ for any (t_i). Suppose the above assertion is false, i.e. for every $\tau\subset\sigma_2$, $|\tau| \geqq \frac{1}{2}n_2$, there is a sequence (b_j), $b_j = 0$ if $j\notin\tau$, such that $\Sigma\, |b_j|^2 = 1$ and $|\Sigma\, b_j y_j|_2 < \frac{1}{4}\alpha$. We construct, by induction, a sequence (τ_i) of subsets of σ_2 and vectors $z_i = \Sigma_{j\in\tau_i} b_{ij} y_j$ to satisfy

(i) $\tau_1 = \sigma_2$,

(ii) $|z_i|_2 \leqq \frac{1}{4}\alpha$, $\Sigma_j\, |b_{ij}|^2 = 1$, $b_{ij} = 0$ if $j\notin\tau_i$,

(iii) $\tau_{k+1} = \{j\in\tau_k : \Sigma_{i\leqq k}\, |b_{ij}|^2 < 1\}$,

as long as we can, i.e. until after, say, m steps $|\tau_{m+1}| < \frac{1}{2}n_2$. Note that our construction implies

(j) $\Sigma_{i,j}\, |b_{ij}|^2 = m$,

(jj) $\Sigma_{i\leqq m}\, |b_{ij}|^2 < 2$ for all j,

(jjj) $\Sigma_{i\leqq m}\, |b_{ij}|^2 \geqq 1$ if $j\in\sigma_2\sim\tau_{m+1}$ and $|\sigma_2\sim\tau_{m+1}| > \frac{1}{2}n_2$.

An easy computation shows now that $m < 2n_2$.

Let (g_i) be a sequence of i.i.d. N(0, 1) Gaussian random variables. We have (E stands for expected value)

$$\mathbf{E}\left|\sum_{i\leqq m} g_iz_i\right|_1 = \mathbf{E}\left|\sum_{j\leqq n_2}\left(\sum_{i\leqq m} b_{ij}g_i\right)y_j\right|_1$$

$$\geqq \alpha\mathbf{E}\sum_j\left|\sum_i b_{ij}g_i\right| - 2^{3/2}\delta n_2^{1/2}\mathbf{E}\left(\sum_j\left|\sum_i b_{ij}g_i\right|^2\right)^{1/2}$$

$$> \alpha(2/\pi)^{1/2}\sum_j\left(\sum_i |b_{ij}|^2\right)^{1/2} - 2^{3/2}\delta n_2^{1/2}\left(\sum_{i,j}|b_{ij}|^2\right)^{1/2}$$

$$\geqq \alpha(2/\pi)^{1/2}\cdot\tfrac{1}{2}n_2 - 2^{3/2}\delta n_2^{1/2}(2n_2)^{1/2}$$

$$= (\alpha/(2\pi)^{1/2} - 4\delta)n_2.$$

On the other hand,

$$\mathbf{E}\left|\sum_{i\leqq m} g_iz_i\right|_1 = \left|\mathbf{E}\left|\sum_i g_iz_i\right|\right|_1 \leqq (2/\pi)^{1/2}\left|\left(\sum_i |z_i|^2\right)^{1/2}\right|_1$$

$$\leqq (2/\pi)^{1/2}\cdot n_2^{1/2}\left|\left(\sum_i |z_i|^2\right)^{1/2}\right|_2 \leqq (2/\pi)^{1/2}n_2^{1/2}\left(\sum_i |z_i|_2^2\right)^{1/2}$$

$$\leqq (2/\pi)^{1/2}n_2^{1/2}\cdot\tfrac{1}{4}\alpha m^{1/2} < \alpha/2\pi^{1/2}n_2,$$

where, for $v\in l_2$, the operations $|v|$ and $|v|^2$ are understood coordinatewise. This is contradictory if δ is chosen small enough (e.g. $\delta=\alpha/8$). Note that the argument proves Lemma B with $c=2^{-10}\alpha^2$.

Lemma C. *Assume $z_1,\dots,z_n\in H$ (a Hilbert space) satisfies $|\Sigma t_iz_i| \geqq (\Sigma|t_i|^2)^{1/2}$ for all scalars (t_i) and that $P: H\to H$ is an orthogonal projection with corank $P\leqq \alpha n$. Then, for every $\delta>4\alpha$, there exists $\sigma\subset\{1,\dots,n\}$, $|\sigma|\geqq(1-\delta)n$ such that, for all scalars $(t_i)_{i\in\sigma}$,*

$$\left|\sum_{i\in\sigma} t_iPz_i\right| \geqq c(\delta)\left(\sum_{i\in\sigma}|t_i|^2\right)^{1/2},$$

where $c(\delta)\geqq\delta^d$, and d is a universal positive constant.

Proof. Without loss of generality we can assume that $H=[z_1,\dots,z_n]=l_2^n$ (since if $P_0=P_{[z_1,\dots,z_n]\cap(\ker P)^\perp}$, then rank $P_0\geqq(1-\alpha)n$ and, for every $x\in H$, $|Px|\geqq|P_0x|$). Moreover, we can assume that $z_j=e_j$ for all j: set T by $Te_j=z_j$ and $P'=T^{-1}PT$, then $\|T^{-1}\|\leqq 1$ and P' is a projection on l_2^n with rank $P'\geqq(1-\alpha)n$ (P' is not necessarily orthogonal); now let $P''=P_{(\ker P')^\perp}$, then, for any scalars (t_j),

$$\left|\sum t_j P z_j\right| = \left|\sum t_j PTe_j\right| = \left|T\left(\sum t_j P'e_j\right)\right| \geqq \| T^{-1} \| \left|\sum t_j P'e_j\right|$$
$$\geqq \left|P'\left(\sum t_j e_j\right)\right| \geqq \left|P''\left(\sum t_j e_j\right)\right|.$$

Hence we need only to show that there exist σ, $c(\delta)$, etc. such that

$$(*) \qquad \left|\sum_{j\in\sigma} t_j P e_j\right| \geqq c(\delta)\left(\sum_{j\in\sigma} |t_j|^2\right)^{1/2},$$

where P is an orthogonal projection, rank $P \geqq (1-\alpha)n$.

We construct σ by an inductive procedure: $\sigma = \sigma_1 \cup \cdots \cup \sigma_k$ where, with the notation $\tau_s = \sigma_1 \cup \cdots \cup \sigma_s$, $|\sigma_s| \geqq c' |\sim \tau_{s-1}|$ and, consequently, the number of steps $k \leqq C' \log 1/\delta$ (c', $C' > 0$ — numerical constants). The procedure is as follows: suppose we have already defined $\sigma_1, \ldots, \sigma_{s-1}$ and that $|\sim \tau_{s-1}| > \delta n \geqq 4\alpha n$. Set $E = [e_i]_{i\in\sim\tau_{s-1}}$, then $\dim E > 4\alpha n$. Let $E'_s = E \cap (\ker P)^{\perp}$; then $\dim E'_s \geqq \dim E - \alpha n$ and so

$$|\{i \in \sim\tau_{s-1} : |P_{E'_s} e_i|^2 \geqq \tfrac{1}{2}\}| \geqq \dim E - 2\alpha n \geqq \tfrac{1}{2}\dim E = \tfrac{1}{2}|\tau_{s-1}|.$$

By Lemma B, there exists $\sigma_s \subset \sim\tau_{s-1}$, $|\sigma_s| \geqq c' |\sim\tau_{s-1}|$, such that, for all scalars $(t_i)_{i\in\sigma_s}$,

$$\left|\sum_i t_i P_{E'_s} e_i\right| \geqq c\left(\sum_i |t_i|^2\right)^{1/2},$$

where $c > 0$ is universal. Consequently, denoting $E''_s = [Pe_i : i \in \tau_{s-1}]^{\perp}$

$$(**) \qquad \left|P_{E''_s}\left(\sum_{i\in\sigma_s} t_i P e_i\right)\right| \geqq c\left(\sum_{i\in\sigma_s} |t_i|^2\right)^{1/2}.$$

The latest implication follows from the following: if $e = \sum_{i\in\sigma_s} t_i e_i$, then

$$|P_{E''_s}(Pe)| \geqq \inf_{h\in\ker P} |P_{E''_s}(e+h)| = \inf_{e'\in E^{\perp},\, h\in\ker P} |Pe' + e + h|$$
$$= \inf_{e'\in E^{\perp},\, h'\in\ker P} |e' + e + h'| \geqq \inf_{h'\in\ker P} |e + P_E h'| = |P_{E'_s} e|.$$

We now show that the obtained σ really satisfies $(*)$. Indeed, $(**)$ implies the existence of mutually orthogonal subspaces $E_1, \ldots, E_k$ of l_2^n ($E_s = E''_s \cap (E''_{s+1})^{\perp}$) such that

(i) $i \in \sigma_s \Rightarrow Pe_i \in [E_1 \cup \cdots \cup E_s]$,

(ii) for all scalars $(t_j)_{j\in\sigma_s}$, $|P_{E_s}(\sum_j t_j P e_j)| \geqq c(\sum_j |t_j|^2)^{1/2}$.

Now, if $x=\Sigma_{j\in\sigma}t_je_j$, denote $x_s'=\Sigma_{j\in\sigma_s}t_je_j$, $y_s'=P_{E_s}x_s'$ and $x_s=x_s'/|x_s'|$, $y_s=y_s'/|y_s'|$. We need

$$\left|\sum_{s\le k}u_sx_s\right|\geqq c(\delta)\left(\sum_{s\le k}|u_s|^2\right)^{1/2}$$

for all scalars (u_s); this is certainly implied by $\|A^{-1}\|\leqq c(\delta)^{-1}$, where $A=(\langle x_s,y_t\rangle)_{s,t\le k}$. Consequently, applying Lemma A and taking into account the fact that $k\leqq C\log 1/\delta$, we get the assertion.

LEMMA D. *Let $D\geqq 0$ and let $x_1,\dots,x_m$ be elements of a Hilbert space H such that $|x_j|\leqq D$ for all j. Then, for any $\alpha>0$, there exists an (orthogonal) projection P on H such that* corank $P<\alpha m$ *and, for any scalars $t_1,\dots,t_m$, $|\Sigma t_jPx_j|\leqq\alpha^{-1/2}D(\Sigma|t_j|^2)^{1/2}$.*

PROOF. Let $T: l_2^m\to H$ be defined by $Te_j=x_j$, $j=1,\dots,m$. Then the Hilbert–Schmidt norm of T is $\leqq m^{1/2}D$ and so the number of eigenvalues of $|T|$ exceeding $\alpha^{1/2}D$ is $<\alpha m$. Choose P to be the spectral projection of TT^* corresponding to the remaining eigenvalues.

The next lemma is the original Dvoretzky–Rogers lemma [D–R].

LEMMA E. *Let $B\subset\mathbf{R}^m$ be a compact convex body symmetric with respect to the origin such that the Euclidean unit ball has the smallest volume of all the ellipsoids containing B. Then there exists an orthonormal basis $u_1,\dots,u_m$ of $\mathbf{R}^m$ and $x_1,\dots,x_m\in B$ such that*

(i) $\langle x_i,u_j\rangle=0$ *if* $1\leqq i<j\leqq m$,

(ii) $\langle x_i,u_i\rangle\geqq((m-i+1)/m)^{1/2}$ *for* $i=1,\dots,m$.

We are now ready to conclude the proof of Proposition 3. Fix $\delta>0$. We first introduce an appropriate Euclidean structure in X, generated by an ellipsoid $\mathscr{E}$ (the unit ball in that structure), to be obtained by a method of "consecutive corrections" with the use of the following construction.

Suppose that $\mathscr{E}_0\subset X$ is an ellipsoid and that $X=E\oplus F$ is an orthogonal (in the $\mathscr{E}_0$-sense) decomposition, $\dim F=m$. Denoting the unit ball of X by B_X, consider $\mathscr{E}'\subset F$ — the ellipsoid of minimal volume containing PB_X (the John ellipsoid), and

$$\mathscr{E}_1=\{sx+ty: s^2+t^2=1, x\in\mathscr{E}_0\cap E, y\in\mathscr{E}'\}.$$

Clearly $\mathscr{E}_1$ is an ellipsoid, $\mathscr{E}_0\cap E=\mathscr{E}_1\cap E$ and E, F are orthogonal also in the $\mathscr{E}_1$-sense; for the rest of the paragraph we are going to work with the Euclidean structure generated by $\mathscr{E}_1$. By Lemma E, there exists an orthonormal basis $u_1, \ldots, u_m$ of F and $x_1, \ldots, x_r \in B_X$, $r > \frac{1}{2}m$, such that $\langle P_F x_j, u_j \rangle \geqq 2^{-1/2}$ for $j = 1, \ldots, r$. Consequently, by Lemma B, there is a subset $\sigma \subset \{1, 2, \ldots, r\}$, $|\sigma| \geqq c_0 r > \frac{1}{2}c_0 m$, such that, for any scalars $(t_j)_{j\in\sigma}$, $|\Sigma_{j\in\sigma} t_j P_F x_j| \geqq 2^{-5/2}(\Sigma_{j\in\sigma}|t_j|^2)^{1/2}$; $c_0 > 0$ is a universal constant. Note that if $F' = [P_F x_1, \ldots, P_F x_r]$, then $P_{F'} B_X \subset \mathscr{E}_1 \cap F'$.

We apply this procedure inductively, denoting the F' obtained in step s by F_s and using, in step s, $E = F_0 + \cdots + F_{s-1}$ (in particular $E = \{0\}$ in step 0), to construct an ellipsoid $\mathscr{E}_2$ and associated with its Euclidean structure in X, an orthogonal decomposition $X = F_0 \oplus F_1 \oplus \cdots \oplus F_k$ with $k \leqq C_0 \log 1/\delta$ (C_0 — a universal constant), $\dim F_k \leqq \frac{1}{2}\delta n$, such that

(i) for each $s = 0, \ldots, k-1$, there is a set

$$\mathscr{X}_s = \{x_1, \ldots, x_p\} \subset B_x \cap F_0 + F_1 + \cdots + F_s,$$

$p = \dim F_s$, satisfying, for all scalars (t_j),

$$\left| \sum_j t_j P_{F_s} x_j \right| \geqq 2^{-5/2} \left(\sum |t_j|^2 \right)^{1/2};$$

(ii) $P_{F_s} B_X \subset \mathscr{E}_2 \cap F_s$ for all s and, consequently, $B_X \subset (k+1)^{1/2}\mathscr{E}_2$.

We now want to show that, for a large subset $\mathscr{X}'$ of $\mathscr{X} = \mathscr{X}_0 \cup \cdots \cup \mathscr{X}_{k-1}$, the lower l_2-estimate still holds. To this end denote, with some abuse of notation, $\mathscr{X} = \{x_j\}$ and apply Lemma D with, say, $\alpha = \delta^{3/2}$ (and $D = (k+1)^{1/2} \leqq (1 + C\log 1/\delta)^{1/2}$). This yields a projection P, corank $P < \delta^{3/2}|\mathscr{X}| < \delta^{3/2} n$, such that, for any scalars (t_j), $|\Sigma_j t_j P x_j| \leqq \delta^{-3/4} D (\Sigma_j |t_j|^2)^{1/2}$. Assuming, as we clearly may, that δ is small, we have in fact $|\Sigma_j t_j P x_j| \leqq \delta^{-1}(\Sigma_j |t_j|^2)^{1/2}$; the same holds of course for any projection Q satisfying $\ker Q \supset \ker P$. Denote, for $s = 0, 1, \ldots, k-1$, $F_s' = F_s \cap (\ker P)^\perp$ and apply Lemma C with $P = P_{F_s'}$ and $\{z_j\} = \mathscr{X}_s$ to obtain

$$\{z_j'\} = \mathscr{X}_s' \subset \mathscr{X}_s, \qquad |\mathscr{X}_s \sim \mathscr{X}_s'| \leqq 4\delta^{3/2}|\mathscr{X}_s| < \tfrac{1}{2}\delta|\mathscr{X}_s|$$

such that, for any scalars $(t_j)_{t\in\mathscr{X}_s'}$,

$$\delta^{-1}\left(\sum |t_j|^2\right)^{1/2} \geqq \left| \sum t_j P_{F_s'} z_j' \right| \geqq \delta^d \left(\sum |t_j|^2 \right)^{1/2}.$$

Set $\mathscr{X}' = \bigcup_{s<k} \mathscr{X}_s'$, then $|\mathscr{X}'| > (1-\delta)n$. The same argument as in the conclusion of the proof of Lemma C shows that the elements of $\mathscr{X}'$ and the

Euclidean norm related to $\mathscr{E}_2$ satisfy the condition (2) from Proposition 3 with $b \geqq \exp(-C_1(\log 1/\delta)^2)$, where $C_1 > 0$ is a universal constant.

It remains to make sure that the condition (1) from Proposition 3 also holds. To this end, let $\mathscr{E}^*$ be an ellipsoid such that $d_X^{-1}\mathscr{E}^* \subset B_X \subset \mathscr{E}^*$ (a distance ellipsoid). Next observe that $\mathscr{E}^* \cap (k+1)^{1/2}\mathscr{E}_2$ is, up to a factor $2^{1/2}$, equivalent to an ellipsoid, i.e. there exists an ellipsoid $\mathscr{E}$ with $2^{-1/2}\mathscr{E} \subset \mathscr{E}^* \cap (k+1)^{1/2}\mathscr{E}_2 \subset \mathscr{E}$. Then the Euclidean structure associated with $\mathscr{E}$ clearly satisfies the condition (1); in (2) we absorb the constants $2^{-1/2}$ and $(k+1)^{-1/2}$ into b, which is not a problem. Q.E.D.

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